STABLE COHOMOLOGY OF POLYVECTOR FIELDS

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Abstract. We show that the stable cohomology of the algebraic polyvector fields on $\mathbb{R}^n$, with values in the adjoint representation is (up to some known classes) the symmetric product space on the cohomology of M. Kontsevich's graph complex.

1. Introduction

Let $T_{\rm poly}^{(n)}$ be the space of algebraic polyvector fields on $\mathbb{R}^n$. Concretely, $T_{\rm poly}^{(n)} = \mathbb{R}[x^1,\dots,x^n,\xi_1,\dots,\xi_n]$ is the space of polynomials in some degree 0 variables $x^1,\dots,x^n$ and degree 1 variables $\xi_1,\dots,\xi_n$. $T_{\rm poly}^{(n)}$ is a Gerstenhaber algebra with the obvious product and the bracket uniquely defined by the relations

$$[\xi_i, x^j] = \delta_i^j.$$

There are inclusions of Gerstenhaber algebras

$$\cdots \to T_{\rm poly}^{(n)} \to T_{\rm poly}^{(n+1)} \to T_{\rm poly}^{(n+2)} \to \cdots .$$

We define

$$T_{\rm poly} = \lim_{\to} T_{\rm poly}^{(n)} = \mathbb{R}[x^1,x^2,\dots,\xi_1,\xi_2,\dots].$$

There is a natural subalgebra $T_{\rm poly}^{\geq 2} \subset T_{\rm poly}$ given by at least quadratic polynomials, i.e., polynomials P such that $P(0)=P'(0)=0$. In the following, we will forget the graded commutative product on $T_{\rm poly}$, we are only interested in the $\mathsf{Lie}\{1\}$ structure.[1] M. Kontsevich has shown the following Theorem:

Theorem 1 (Kontsevich [4])**.** *The $\mathsf{Lie}\{1\}$ algebra cohomology of $T_{\rm poly}^{\geq 2}$ with values in the trivial representation is given by*

$$H(T_{\rm poly}^{\geq 2}, \mathbb{R}) = \mathbf{S}(H_P(\mathfrak{gl},\mathbb{R}) \oplus H(\mathsf{GC}))$$

Here $H_P(\mathfrak{gl})$ is the primitive stable cohomology of the general linear $\mathsf{Lie}\{1\}$ algebras (see below), $\mathbf{S}(\cdot)$ is the completed symmetric product space and GC is M. Kontsevich's graph complex (see, e.g. [10] or [5]).

Remark. Concretely the primitive stable cohomology $H_P(\mathfrak{gl},\mathbb{R})$ is

$$H_P(\mathfrak{gl},\mathbb{R}) = \prod_{n=1,5,9,\dots} \mathbb{R} w_n$$

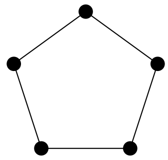
where the cycles w_n are defined as follows:

$$w_n(\gamma_1,\dots,\gamma_n) := \begin{cases} \sum_{\sigma\in\mathbb{S}_n} tr(\gamma_{\sigma_1}\gamma_{\sigma_2}\cdots\gamma_{\sigma_n}) & \text{for } \gamma_1,\dots,\gamma_n \in \mathfrak{gl} \subset T_{\rm poly} \\ 0 & \text{otherwise} \end{cases}$$

Key words and phrases. Formality, Deformation Quantization.

[1] Here $\mathsf{Lie}\{1\}$ is the degree shifted Lie operad. Defining a $\mathsf{Lie}\{1\}$ structure on a space V is equivalent to defining a Lie structure on the degree shifted space $V[1]$. The reader who likes Lie algebras better than $\mathsf{Lie}\{1\}$ algebras may replace all occurrences of $\mathsf{Lie}\{1\}$ by Lie and all occurrences of $T_{\rm poly}$ by $T_{\rm poly}[1]$. We work with $\mathsf{Lie}\{1\}$ here since it is more convenient regarding signs.

Here $\mathbb{S}_n$ is the symmetric group on n symbols. The cycles w_n may also be identified with "wheel" graphs in a suitable graph complex, for example:

$w_5 \leftrightarrow$ 

Our goal here is to extend Theorem 1 to Lie algebra cohomology of T_{poly} with values in the adjoint representation. One remark is in order. There are three complexes which one may understand as the complex computing the stable cohomology of the $T_{\mathrm{poly}}^{(n)}$, namely:

- The Chevalley complex $C(T_{\mathrm{poly}}, T_{\mathrm{poly}})$.
- The Chevalley complex $C(T_{\mathrm{poly}}, \hat{T}_{\mathrm{poly}})$, where
$$\hat{T}_{\mathrm{poly}} = \mathbb{R}[[x^1, x^2, \dots, \xi_1, \xi_2, \dots]]$$
is the completed version of T_{poly} with respect to the natural grading by degree.
- The Chevalley complex $C(T_{\mathrm{poly}}, \tilde{T}_{\mathrm{poly}})$, where
$$\tilde{T}_{\mathrm{poly}} = \lim_{\leftarrow} T_{\mathrm{poly}}^{(n)}$$
is the completed version of T_{poly} with respect to the natural filtration by dimension.

The latter complex can be seen as
$$\lim_{\leftarrow} C(T_{\mathrm{poly}}^{(n)}, T_{\mathrm{poly}}^{(n)}).$$
Here the maps
$$C(T_{\mathrm{poly}}^{(n)}, T_{\mathrm{poly}}^{(n)}) \leftarrow C(T_{\mathrm{poly}}^{(n+1)}, T_{\mathrm{poly}}^{(n+1)})$$
are defined by using (i) the embedding $T_{\mathrm{poly}}^{(n)} \to T_{\mathrm{poly}}^{(n+1)}$ from above and (ii) the map $T_{\mathrm{poly}}^{(n+1)} \to T_{\mathrm{poly}}^{(n)}$ of $T_{\mathrm{poly}}^{(n)}$-modules. Note that we have the inclusions
$$C(T_{\mathrm{poly}}, T_{\mathrm{poly}}) \subset C(T_{\mathrm{poly}}, \tilde{T}_{\mathrm{poly}}) \subset C(T_{\mathrm{poly}}, \hat{T}_{\mathrm{poly}}).$$

The result of this paper is that the cohomology of the second and third version is as follows.

Theorem 2. *The* $\mathsf{Lie}\{1\}$ *algebra cohomology of* T_{poly} *with values in* $\tilde{T}_{\mathrm{poly}}$ *or* $\hat{T}_{\mathrm{poly}}$ *is*
$$H(T_{\mathrm{poly}}, \hat{T}_{\mathrm{poly}}) = H(T_{\mathrm{poly}}, \tilde{T}_{\mathrm{poly}}) = \mathbf{S}(\mathbb{R}S \oplus \prod_{n=1,5,9,\dots} \mathbb{R}W_n \oplus H(\mathsf{GC})).$$
Here W_n *denotes to the cocycle corresponding to a wheel graph with* n *vertices (see below),* $\mathbf{S}(\cdot)$ *is the completed symmetric product space and* GC *is again M. Kontsevich's graph complex. The additional generator* S *acts on a homogeneous polynomial* $p \in \mathbb{R}[x^1, x^2, \dots, \xi_1, \xi_2, \dots]$ *of* (x, ξ)*-degree* (r_1, r_2) *as* $S(p) = (r_1 + r_2 - 2)p$.

The cohomology of $C(T_{\mathrm{poly}}, T_{\mathrm{poly}})$ is more subtle, so we ban its treatment to section 6 below. The definition of M. Kontsevich's graph complex will be recalled in section 2. There it will also be recalled how graph cocycles can be mapped to $\mathsf{Lie}\{1\}$ algebra cocycles.

Remark. For simplicity we use $\mathbb{R}$ as our ground field. However, one may replace $\mathbb{R}$ by any field of characteristic zero.

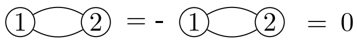
Acknowledgements. This work benefitted from discussions with A. Khoroshkin and V. Dolgushev. In particular I am grateful to V. Dolgushev for sharing an early version of his manuscript [3] with me. Section 7 is dedicated to clarifying the relation of the present work with his.

2. Recollection: M. Kontsevich's graph complex

We recall here the definition of M. Kontsevich's graph complex [5], following [10] and [9].[2] Let us define the spaces

$$\mathsf{Gra}(n) := \prod_k \left(\{\mathbb{R}\text{-linear comb. of graphs with vertex set } n \text{ and edge set } k\} \otimes \mathbb{R}[k]\right)_{\mathbb{S}_k}.$$

Here the notation is as follows. A *graph* with vertex set $[n] := \{1, \ldots, n\}$ and edge set k is an ordered set of unordered pairs of elements in $[n]$. We understand the k-th pair in this set as the k-th edge, or the edge labelled by k. Note that in particular, we allow *tadpoles* or *short cycles*, i.e., edges of the form (i, i). The symmetric group S_k acts on such a graph by permuting the labels on edges. In the definition of $\mathsf{Gra}(n)$ we let $\mathbb{S}_k$ act also on $\mathbb{R}[k]$ by sign. Hence each edge contributes -1 to the degree, and exchanging the labels on two edges produces a minus sign. For example, the following element of $\mathsf{Gra}(2)$ is zero by symmetry:

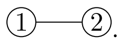


In drawings we generally suppress the edge labels for simplicity, thus leaving a sign ambiguity. The vector space $\mathsf{Gra}(n)$ carries a natural right action of $\mathbb{S}_n$ by permuting the vertices. In fact, the spaces $\mathsf{Gra}(n)$ assemble to form an operad Gra. The operadic composition is given by "inserting" one graph at a vertex of another graph (see [10, 9]). There is a natural action of Gra on T_{poly}. A graph $\Gamma \in \mathsf{Gra}(n)$ acts on polyvector fields $\gamma_1, \ldots, \gamma_n \in T_{\mathrm{poly}}$ by the formula

$$\Gamma(\gamma_1, \ldots, \gamma_n) = \mu \circ \left(\prod_{(i,j)} \sum_{k=1}^{d} \frac{\partial}{\partial x^k_{(j)}} \frac{\partial}{\partial \xi^{(i)}_k} + \frac{\partial}{\partial x^k_{(i)}} \frac{\partial}{\partial \xi^{(j)}_k} \right) (\gamma_1 \otimes \cdots \otimes \gamma_n).$$

Here μ is the operation of multiplication of n polyvector fields and the product runs over all edges (i, j) in Γ, in the order given by the numbering of edges. The notation $\frac{\partial}{\partial x^k_{(j)}}$ means that the partial derivative is to be applied to the j-th factor of the tensor product, and similarly for $\frac{\partial}{\partial \xi^{(i)}_k}$.

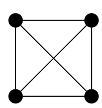
There is a map of operads $\mathsf{Lie}\{1\} \to \mathsf{Gra}$, sending the generator to the graph



The full graph complex is defined as the deformation complex

$$\mathsf{fGC} := \mathrm{Def}((\mathsf{Lie}\{1\})_\infty \to \mathsf{Gra}) \cong \prod_{n=1}^{\infty} \mathsf{Gra}(n)^{\mathbb{S}_n}[2-2n].$$

Elements are linear combinations of graphs, symmetric under permutations of the vertex numbers. In pictures, we indicate this by drawing black vertices without numbers:



[2]The notation here will simplified compared to that from [10]. In particular the graph complex GC here corresponds to GC_2 there, and fGC here corresponds to $\mathsf{fGC}^{\circlearrowleft}$ there.

fGC is a dg lie algebra, as is any deformation complex. The differential is the bracket with the Maurer-Cartan element

$$\bullet\!\!-\!\!\!-\!\!\!-\!\!\bullet.$$

This amounts to splitting vertices of graphs.

Definition 1. *M. Kontsevich's graph complex* GC *is the sub-dg Lie algebra of* fGC *spanned by connected graphs without tadpoles (short cycles) all of whose vertices are at least trivalent.*

It is shown in [10] that this is indeed a sub dg lie algebra. The graph cohomology is the cohomology of GC and is in general hard to compute. The main result of [10] (and [6]) is the following Theorem:

Theorem 3 ([10], [6])**.** $H^0(\mathsf{GC}) \cong \mathfrak{grt}$ *is the Grothendieck Teichmüller Lie algebra. Furthermore* $H^{<0}(\mathsf{GC}) = 0$*.*

It is not known how to compute the higher graph cohomology. It is however known that it is not zero by computer experiments, see e.g. [2]. Because of the action of Gra on T_{poly} there is a natural map from fGC into the Chevalley complex $C(T_{\mathrm{poly}}, T_{\mathrm{poly}})$ of T_{poly}:

$$\mathsf{fGC} = \mathrm{Def}((\mathsf{Lie}\{1\})_\infty \to \mathsf{Gra}) \to \mathrm{Def}((\mathsf{Lie}\{1\})_\infty \to \mathrm{End}(T_{\mathrm{poly}})) \to C(T_{\mathrm{poly}}, T_{\mathrm{poly}}).$$

This explains how graph cohomology classes can be understood as $\mathsf{Lie}\{1\}$ algebra cohomology classes (with values in the adjoint representation) in Theorem 2. It does not yet explain how graph cohomology classes can be understood as $\mathsf{Lie}\{1\}$ algebra cohomology classes of $T_{\mathrm{poly}}^{\geq 2}$ with values in the trivial representation. However, note that by the inclusion $T_{\mathrm{poly}}^{\geq 2} \to T_{\mathrm{poly}}$ and by the map of $T_{\mathrm{poly}}^{\geq 2}$-modules $T_{\mathrm{poly}} \to \mathbb{R}$ there is a natural map of complexes

$$C(T_{\mathrm{poly}}, T_{\mathrm{poly}}) \to C(T_{\mathrm{poly}}^{\geq 2}, T_{\mathrm{poly}}) \to C(T_{\mathrm{poly}}, \mathbb{R}).$$

This means that graph cohomology classes also yield $\mathsf{Lie}\{1\}$ algebra cohomology classes of $T_{\mathrm{poly}}^{\geq 2}$ with values in the trivial representation, as in Theorem 1.

2.1. **Directed graphs.** Instead of working with undirected graphs as above, one may work with directed graphs. The definitions from above carry over to this setting. In particular, we obtain an operad dGra of directed graphs, and a directed full graph complex

$$\mathsf{dfGC} = \mathrm{Def}((\mathsf{Lie}\{1\})_\infty \to \mathsf{dGra}).$$

It also acts onT_{poly} in a natural way. There is a map of operads

$$\mathsf{Gra} \to \mathsf{dGra}$$

sending an undirected graph to the sum of directed graphs obtained by assigning arbitrary directions to the edges. Hence we also have a map of dg Lie algebras $\mathsf{fGC} \to \mathsf{dfGC}$. We make the following definition

Definition 2. *The directed graph complex* dGC *is the sub-dg Lie algebra of* dfGC *spanned by connected graphs all of whose vertices are at least two-valent.*

Note that here we require vertices to be at least two-valent instead of trivalent. There is a map of dg Lie algebras $\mathsf{GC} \to \mathsf{dGC}$. The following result was shown in [10].[3]

[3] A significantly weaker statement can also be found in [1].

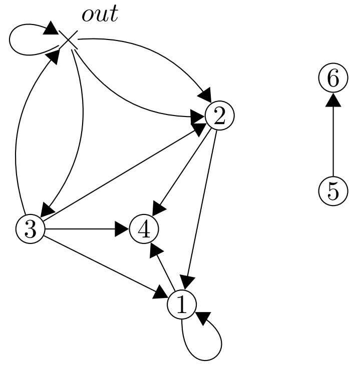
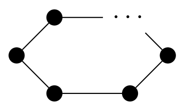


Figure 1. Some xgraph. It has three connected components after deleting *out*, namely: (i) the tadpole at *out*, (ii) the two vertices on the right and (iii) the remainder of the graph.

Proposition 1. *[see* [10]*]*

$$H(\mathsf{dGC}) \cong \prod_{n=1,5,9,\dots} \mathbb{R}W_n \oplus H(\mathsf{GC}).$$

Here W_n corresponds to the wheel graph with n vertices:



Remark. This means in particular that the cohomology of all the graph complexes above can be expressed using $H(\mathsf{GC})$, up to some known classes.

3. Preliminary definition: xgraphs

Definition 3. *An* xgraph *is a directed graph with vertex set* $\{out\} \sqcup [n]$ *(here* $[n] = \{1, \dots, n\}$ *and* n *can be* $0, 1, 2, 3, \dots$*). The special vertex "out" we call the output vertex, the vertices* $[n]$ *we call the input vertices. The edges are labelled by numbers* $1, 2, \dots, k$*, i.e., the edge set is* $[k]$*.*

Note in particular that tadpoles are allowed. They are also allowed at the special vertex *out*. A typical xgraph is shown in Figure 1. Note that we again suppress the edge labels to not clutter the picture too much. We distinguish three kinds of edges: (i) "Normal edges" between two input vertices. They will be considered to be of degree -1. (ii) "Special edges" between one input and the output vertex. They are assigned degree zero. (iii) "Special tadpoles", which are tadpoles at *out*. They will be assigned degree +1.

We denote the number of those edges by k_1, k_2, k_3, i.e., $k = k_1 + k_2 + k_3$. There is a natural action of the symmetric groups $\mathbb{S}_n$ and $\mathbb{S}_k$ on any xgraph with n input vertices and k edges. We define the following vector spaces of xgraphs:

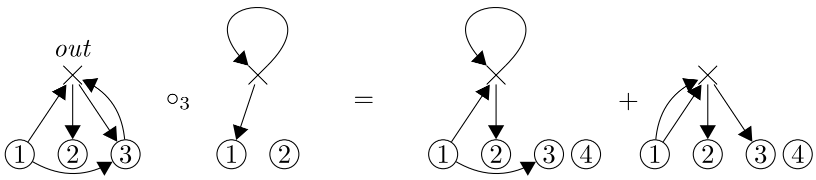
$$\mathsf{XGra}(n) :=$$
$$\prod_k \left(\{\mathbb{R}\text{-linear comb. of xgraphs with } n \text{ input vert. and } k \text{ edges}\}[k_1 - k_3]\right)_{\mathbb{S}_k}.$$

Here the action of $\mathbb{S}_k$ is by permutation of edge labels, with appropriate minus signs if odd edges are interchanged.

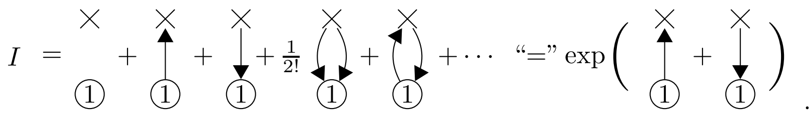
The spaces $\mathsf{XGra}(n)$ assemble to form an operad XGra. Let Γ, Γ' be xgraphs then the composition $\Gamma \circ_j \Gamma'$ is defined as follows:

(1) If the number of incoming edges at vertex j of Γ is not equal to the number of outgoing edges at *out* of Γ' the result is zero. If the number of outgoing edges at vertex j of Γ is not equal to the number of incoming edges at *out* of Γ' the result is also zero. Otherwise proceed.
(2) Delete the input vertex j of Γ and vertex *out* of Γ'. this leaves several "dangling" edges in Γ and Γ'.
(3) The result is the sum over all possible xgraphs obtained by gluing outgoing dangling edges of Γ to incoming dangling edges of Γ' and vice versa. Here the numbering on edges is modified such that edges of Γ have smaller labels than those of Γ'.

Here is an example of an operadic composition:



The operad dGra from above acts from the right on (the $\mathbb{S}$-module) XGra by "insertions". Furthermore there is a natural map of operads $\mathsf{dGra} \to \mathsf{XGra}$ by right action on the identity element $I \in \mathsf{XGra}(1)$. Concretely

(1)


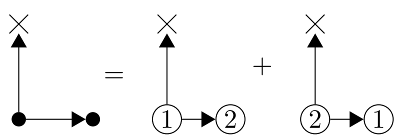
The sum is over all possible ways to connect *out* to one vertex. The coefficient of each term is $\frac{1}{k!l!}$ where k and l are the numbers of in- and outgoing edges at *out*. From the map $\mathsf{Lie}\{1\} \to \mathsf{dGra}$ we hence get a map $\mathsf{Lie}\{1\} \to \mathsf{XGra}$.

Definition 4. *The* full extended graph complex *is the deformation complex*

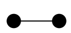
$$\mathsf{fXGC} = \mathrm{Def}(\mathsf{Lie}\{1\} \to \mathsf{XGra}).$$

The extended graph complex *is the subcomplex* $\mathsf{XGC} \subset \mathsf{fXGC}$ *spanned by graphs which are connected and non-empty after deleting the vertex out.*[4]

fXGC is a dg Lie algebra (as is any deformation complex). XGC is a Lie subalgebra. Note that from the inclusions of operads $\mathsf{Gra} \to \mathsf{dGra} \to \mathsf{XGra}$ we obtain inclusions of dg Lie algebras $\mathsf{GC} \to \mathsf{dGC} \to \mathsf{XGC}$. Elements of $\mathsf{fXGC}(n)$ are series in xgraphs (modulo permutations of the edge labels), symmetric under interchange of the vertex labels. We indicate this in pictures by not drawing vertex labels but black dots instead, understanding that one should sum over all possible ways of putting vertex labels. For example

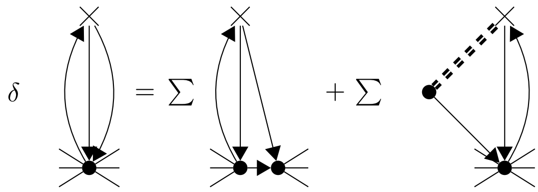


Let us discuss the differential on XGC (and fXGC). It is defined as the bracket with the element

$m = I \circ$  .

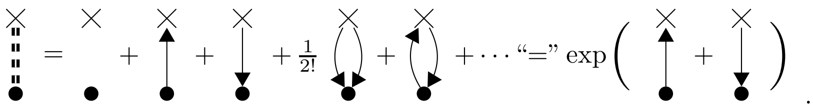
[4]The non-emptiness condition excludes the graph consisting of just the vertex *out*.

The bracket produces two terms, which one can depict as follows.



(2)

Here only one vertex and the output vertex are drawn, it should be understood that this is only a part of the bigger graph, and the differential acts on the other parts similarly. To produce the first term on the right the vertex is split into two and the incoming edges are reconnected in an arbitrary way. To produce the second term one detaches one edge from *out* and connects it to a newly created vertex. One furthermore sums over all ways of connecting this new vertex to *out*. In other words, the double dashed line stands for



Note that the first term in particular produces a new vertex of valence one. This term will be important later.

3.1. **An alternative definition of XGra and fXGC.** Consider the endomorphism operad of $T_{\rm poly}^{(n)}$, i.e., $\mathrm{End}(T_{\rm poly}^{(n)})$. It carries a natural action of $GL(n)$. Let us consider the space of $GL(n)$ invariants $\mathrm{End}(T_{\rm poly}^{(n)})^{GL(n)}$. $\mathrm{End}(T_{\rm poly}^{(n)})$ is made of products and symmetric products of the fundamental representation $\mathbb{R}^n$ and its dual $(\mathbb{R}^n)^*$. Hence, by classical invariant theory elements of $\mathrm{End}(T_{\rm poly}^{(n)})^{GL(n)}$ can be written as certain graphs, where each edge stands for one pairing of $\mathbb{R}^n$ and $(\mathbb{R}^n)^*$. If one looks at the structure of the $GL(n)$ representation $\mathrm{End}(T_{\rm poly}^{(n)})$, one sees that these graphs are xgraphs as defined above (apart from the labelling on edges). The operadic composition on XGra has been defined so that it agrees with the operadic composition on $\mathrm{End}(T_{\rm poly}^{(n)})^{GL(n)}$.

Note that for finite n not all xgraphs yield nonzero elements of $\mathrm{End}(T_{\rm poly}^{(n)})^{GL(n)}$. Instead

$$\mathrm{End}(T_{\rm poly}^{(n)})^{GL(n)} \cong \mathsf{XGra}/I_n$$

where I_n is an ideal spanned by (linear combinations of) xgraphs acting as zero. Note that $\lim_{\to} I_n = 0$. Hence we arrive at the following alternative definition of XGra.

Alternative Definition:

$$\mathsf{XGra} = \lim_{\leftarrow} \mathrm{End}(T_{\rm poly}^{(n)})^{GL(n)}.$$

From this alternative definition is is also clear that there is a map of operads $\mathsf{dGra} \to \mathsf{XGra}$, because dGra acts on $T_{\rm poly}$ in a $GL(\infty)$ invariant way.

3.2. **The action on polyvector fields.** The action of the operad dGra on $T_{\rm poly}^{(n)}$ factors through $\mathsf{dGra} \to \mathsf{XGra}$. The action of XGra on $T_{\rm poly}^{(n)}$ is clear from the alternative definition in the previous subsection. Let us nevertheless describe it combinatorially for completeness. The action of an xgraph $\Gamma \in \mathsf{XGra}$ with N input vertices on polyvector fields $\gamma_1, \dots, \gamma_N \in T_{\rm poly}^{(n)}$ is given as follows.

(1) If Γ has a special tadpole, then

$$\Gamma(\gamma_1, \dots, \gamma_N) = \pm \left(\sum_{j=1}^{n} x^j \xi_j \right) \wedge \Gamma'(\gamma_1, \dots, \gamma_N)$$

where Γ' is obtained from Γ by deleting the special tadpole. The sign is "+" if the special tadpole is the first in the ordering of edges. So the special tadpole indicates multiplication with the Euler vector field.

(2) Suppose Γ has no special tadpole. Suppose also that the $\gamma_1, \dots, \gamma_N$ are of homogeneous degree in the x variables and ξ variables separately. Say γ_j has x degree k_j and ξ degree l_j. Let $\tilde{\Gamma}$ be the graph in dGra obtained by deleting the vertex *out* and all adjacent edges. Then

$$\Gamma(\gamma_1, \dots, \gamma_N) = \begin{cases} \left(\prod_{j=1}^{N} k_j! l_j! \right) \tilde{\Gamma}(\gamma_1, \dots, \gamma_N) & \text{if condition } (*) \text{ below is satisfied} \\ 0 & \text{otherwise} \end{cases}$$

Here condition $(*)$ is that the number of outgoing edges at vertex j is l_j and the number of incoming edges is k_j for all $j = 1, 2, \dots, N$.

By this action we automatically obtain a map of dg Lie algebras

$$\mathsf{XGC} \to C(T^{(n)}_{\mathrm{poly}}, T^{(n)}_{\mathrm{poly}}).$$

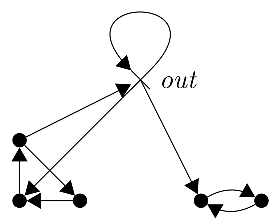
Note that XGra in our definition does not act on the stable version T_{poly} because the Euler vector field is not contained in it. However, the suboperad $\mathsf{XGra}^{nt} \subset \mathsf{XGra}$ of xgraphs without special tadpoles acts on T_{poly}. In particular, we get a map of dg Lie algebras

$$\mathsf{XGC}^{nt} \to C(T_{\mathrm{poly}}, T_{\mathrm{poly}}).$$

where $\mathsf{XGC}^{nt} \subset \mathsf{XGC}$ is spanned by graphs without special tadpoles.

4. The cohomology of fXGC and XGC

For an xgraph Γ we define its set of connected components as the set of connected components of the graph obtained by deleting the vertex *out*. For example, the following graph has three connected components in this sense.



In particular, a special tadpole counts as one connected component. Note that the differential on fXGC leaves invariant the connected components.Hence the following result is immediate.

Lemma 1.

$$H(\mathsf{fXGC}) \cong \mathbf{S}(H(\mathsf{XGC})).$$

So let us focus on XGC. The goal of this section is to show the following result.

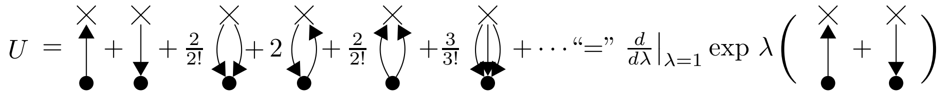
Proposition 2.

$$H(\mathsf{XGC}) \cong H(\mathsf{dGC}) \oplus \mathbb{R}S.$$

Here the inclusion $\mathsf{dGC} \to \mathsf{XGC}$ appeared already in section 3. The last element S is defined as

$$S = U - 2I$$

where $I \in \mathsf{XGra}(1)$ is the unit element (see (1)) and U is the following series of graphs:



in $\mathsf{XGra}(1)$. The operation U corresponds to the scaling operator, acting on multivector fields of degree (joint in x's and ξ's) k as multiplication by k. Let us prove the proposition above. In fact, we will prove a somewhat stronger statement.

Claim 1: The inclusion $\mathsf{dGC} \oplus \mathbb{R}S \to \mathsf{XGC}$ is a quasi-isomorphism.

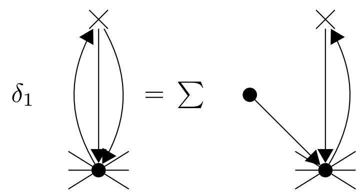
To prove the claim, consider the descending complete filtration on XGC by the valence of *out*.

$$\mathsf{XGC} = \mathcal{F}^0 \supset \mathcal{F}^1 \supset \mathcal{F}^2 \supset \cdots$$

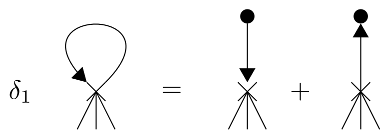
where $\mathcal{F}^p$ is spanned by graphs for which *out* has valence $\geq p$. The filtration is not compatible with the differential in the usual sense, but $\delta \mathcal{F}^p \subset \mathcal{F}^{p-1}$. Consider the associated graded. The leading part of the differential is

$$\delta_1 : \mathcal{F}^p / \mathcal{F}^{p+1} \to \mathcal{F}^{p-1} / \mathcal{F}^p.$$

This differential reduces the valence of *out* by one. Considering "equation" (2), we see that the only part of δ that can achieve this is the following:



So δ_1 detaches one special edge from *out* and connects it to a new vertex. Note that the same happens for special tadpoles:



Concretely, δ_1 has the form (on a graph Γ)

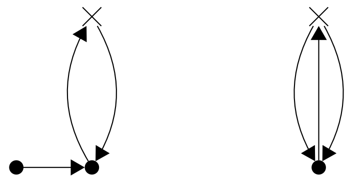
$$\delta_1 \Gamma = \sum_e \Gamma_e.$$

Here the sum runs over all special edges e. Say e connects some vertex $v \neq$ *out* to *out* (*out* to v). Then Γ_e is obtained by (i) adding a new vertex v' to Γ and (ii) reconnect the edge e so as to connect v to v' (v' to v). Note that the reconnected edge has degree -1, while previously it had degree 0. The edge e becomes the first in the list of edges (all other edge labels are shifted by one). For $v =$ *out*, i.e., special tadpoles, the contribution consists of two terms as depicted above.

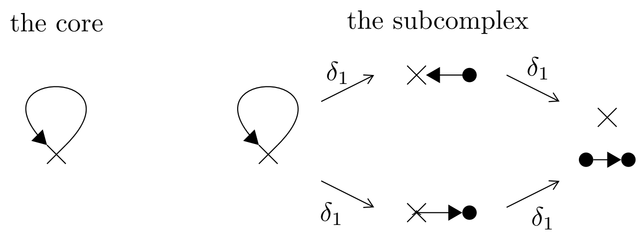
Claim 2: The cohomology of the associated graded with differential δ_1 is

$$H(gr\mathsf{XGC}, \delta_1) = gr^0\mathsf{XGC} / \delta_1 gr^1 \mathsf{XGC} = \mathcal{F}^0 / (\mathcal{F}^1 + \delta_1 \mathcal{F}^1)$$

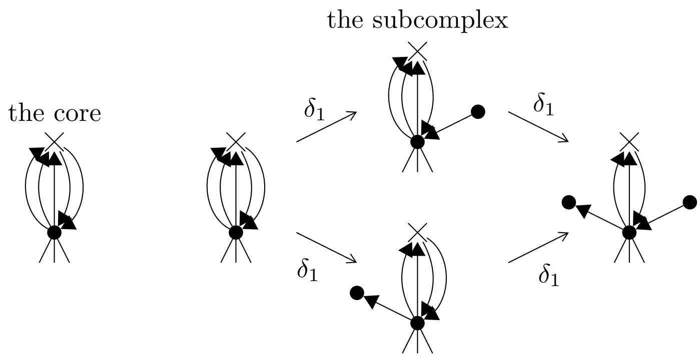
To see this one uses a combinatorial argument along the lines of the proof of Proposition 3 in [10], see also [4]. Let the *core* of an xgraph be the isomorphism class of xgraphs obtained by identifying all valence 1 vertices with *out*.[5] Here is an example of an xgraph (left, labels not shown) and its core (right).



The differential δ_1 cannot change the core. I.e., if it acts on a graph Γ, the result is a sum of graphs with the same core. It follows that $(gr\mathsf{XGC}, \delta_1)$ splits into subcomplexes of graphs having the same cores. Fix one such core. If the core has no input vertex, there are only two possibilities (by connectedness): (i) the core is the "empty graph" consisting of only the vertex *out*, or (ii) the core consists of the vertex *out* and one tadpole at *out*. For case (i) there is (potentially) only one graph with that core, namely the empty graph itself. But this graph is excluded from XGC by definition. For case (ii) the subcomplex corresponding to that core is four dimensional. The four graphs generating it are depicted below. It is clear that the resulting complex is acyclic.



Next consider a core with at least one input vertex. Let us consider the subcomplex corresponding to this core. Focus on one vertex of the core, that is connected by k incoming edges and l outgoing edges to *out*. If $k, l > 0$ graphs with this fixed core locally have one of the following four shapes:



In case $k = 0$ the top right two graphs are missing, if $l = 0$ the bottom right two graphs are missing, and if $k = l = 0$ the three rightmost graphs are absent from this picture.

Now the subcomplex of xgraphs with a fixed core has the form

$$(\otimes_v C_v)^G$$

[5] An isomorphism of an xgraph is a permutation of the edge and vertex labels that maps the edge set to itself.

where G is the symmetry group of the core. The tensor product runs over input vertices in (one representative of) the core and the C_v are relatively simple complexes associated to the vertices. Concretely C_v is of dimension 1,2 or 4, corresponding to the up to 4 graphs occurring in the above drawing. Is is clear that C_v is acyclic if the vertex v in the core has special edges (i.e., edges to *out*) attached to it. In case v has no such edges, c_v is one dimensional, hence the cohomology is also one dimensional. Hence only graphs without special edges can contribute nontrivial cohomology classes of $(gr\mathsf{XGC}, \delta_1)$. This shows Claim 2.

Finally let us deduce Claim 1 from Claim 2. From Claim 2 and the completeness of the filtration it follows that the projection $\mathsf{XGC} \to gr^0\mathsf{XGC}/\delta_1 gr^1\mathsf{XGC}$ is a quasi-isomorphism. By what is said above, the space $\delta_1 gr^1\mathsf{XGC}$ is composed of all graphs in $gr^0\mathsf{XGC}$ that have valence 1 vertices. Hence the composition of maps of complexes

$$\mathsf{dGC} \oplus \mathbb{R}S \to \mathsf{XGC} \to gr^0\mathsf{XGC}/\delta_1 gr^1\mathsf{XGC}$$

is an isomorphism. This shows Claim 1 and the Proposition. $\square$

5. The proof of Theorem 2

The proof of Theorem 2 is more or less a copy of M. Kontsevich's proof of Theorem 1, plus one additional combinatiorial step. Note that the general linear Lie (or $\mathsf{Lie}\{1\}$) algebras act on the Chevalley complex $C(T_{\mathrm{poly}}, \hat{T}_{\mathrm{poly}})$ of T_{poly}, and on the sub-complexes $C(T_{\mathrm{poly}}, \tilde{T}_{\mathrm{poly}})$, $C(T_{\mathrm{poly}}, T_{\mathrm{poly}})$. Let us focus on the first of the three spaces to begin with. It is a projective limit of spaces

$$V_n := C(T^{(n)}_{\mathrm{poly}}, \hat{T}^{(n)}_{\mathrm{poly}})$$

on which $GL(n)$ acts in a reductive manner. Furthermore the action of $\mathfrak{gl}_n$ is trivial on cohomology since it is given by a Cartan type formula ($L_x = d \circ \iota_x + \iota_x \circ d$, where d is the Chevalley differential, $x \in \mathfrak{gl}_\infty$, L_x is the action of x and ι_x is the "insertion operation"). It follows that the cohomology of V_n is the same as that of its invariant part $H(V_n) = H(V_n^{GL(n)}) = H(V_n)^{GL(n)}$. By classical invariant theory the elements in $V_n^{GL(n)}$ can be identified with certain graphs, which can be checked to be xgraphs, see also section 3.1. Hence V_n is quasi-isomorphic to a certain quotient of fXGC. It is a quotient, not the full space fXGC, since due to finite dimensions some graphs act as zero. Let us call this quotient $\mathsf{fXGC}^{(n)}$. We know that:

(1) The maps $V_{n+1} \to V_n$ are surjective.
(2) The maps $\mathsf{fXGC}^{(n)} \to V_n$ are quasi-isomorphisms.
(3) The maps $\mathsf{fXGC}^{(n+1)} \to \mathsf{fXGC}^{(n)}$ are surjective.

By the second assertion the mapping cones $MC(\mathsf{fXGC}^{(n)} \to V_n)$ are acyclic for all n. By assertions 1 and 3 the mapping cones form a tower, with all morphisms surjective:

$$\cdots \leftarrow MC(\mathsf{fXGC}^{(n)} \to V_n) \leftarrow MC(\mathsf{fXGC}^{(n+1)} \to V_{n+1}) \leftarrow \cdots .$$

Hence by [8], Theorem 3.5.8 we can conclude that

$$\begin{aligned} 0 = \lim_{\leftarrow} H(MC(\mathsf{fXGC}^{(n)} \to V_n)) &= H(\lim_{\leftarrow} MC(\mathsf{fXGC}^{(n)} \to V_n)) \\ &= H(MC(\lim_{\leftarrow} \mathsf{fXGC}^{(n)} \to \lim_{\leftarrow} V_n)). \end{aligned}$$

In other words the map

$$\lim_{\leftarrow} \mathsf{fXGC}^{(n)} \to C(T_{\mathrm{poly}}, \hat{T}_{\mathrm{poly}})$$

is a quasi-isomorphism. But $\lim_{\leftarrow} \mathsf{fXGC}^{(n)}$ is easily checked to be fXGC. (Any xgraph acts non-trivially on some $T_{\mathrm{poly}}^{(n)}$ for high enough n. The n depends on the xgraph.) Hence one of the three equalities of Theorem 2 then follows from Lemma 1 and Propositions 2 and 1.

For the sub-complex $C(T_{\mathrm{poly}}, \tilde{T}_{\mathrm{poly}})$ the same argument goes through, one just has to replace $\hat{T}_{\mathrm{poly}}^{(n)}$ by $T_{\mathrm{poly}}^{(n)}$. □

6. The cohomology of T_{poly}.

Let us consider the complex $C(T_{\mathrm{poly}}, T_{\mathrm{poly}})$. Unfortunately, its cohomology is more subtle than that occurring in Theorem 2. For example, consider a sequence of polynomials $p_1, p_2, \dots \in T_{\mathrm{poly}}$ with $p_j \in \mathbb{R}[x_j, \dots, \xi_j, \dots]$ for $j = 1, 2, \dots$. Then the assignment

$$\gamma \mapsto \sum_j [p_j, \gamma]$$

is well defined (only finitely many terms in the sum are nonzero for fixed γ) and defines a 1-cocycle in $C^1(T_{\mathrm{poly}}, T_{\mathrm{poly}})$. It in general represents a non-trivial cohomology class if the sequence $\{p_n\}_n$ is infinite. Another sequence $\{p'_n\}_n$ of the above form gives rise to the same cohomology class if $\{p_n\}_n$ and $\{p'_n\}_n$ eventually agree, i.e., if there is an N such that $p_n = p'_n$ for $n > N$. In particular, the cohomology classes of this form are invariant under the action of $GL(\infty)$, though there is (in general) no $GL(\infty)$ invariant representative. Note however that two sequences can give rise to the same cohomology class even if they do not eventually agree. For example, one may shift sequences or add and subtract the same term from some elements of the sequence.

Remark. Note also that the product of two cohomolgy classes of the above form is trivial. Concretely, for sequences $\{p_n\}_n$ and $\{q_n\}_n$ as above (of homogeneous degrees) the 2-cocyle

$$(\gamma, \nu) \mapsto \sum_{i,j} (-1)^{(|q_j|+1)|\gamma|} [p_i, \gamma] [q_j, \nu] + \sum_{i,j} (-1)^{(|q_j|+1+|\gamma|)|\nu|} [p_i, \nu] [q_j, \gamma]$$

is the coboundary (up to global sign) of the cochain

$$\gamma \mapsto \sum_{i \leq j} p_i [q_j, \gamma] - (-1)^{|q_j||p_i|+|p_i|+|q_j|} \sum_{i>j} q_j [p_i, \gamma] \ .$$

Example 1. The operation

$$T \colon \gamma \mapsto \sum_j \left[x^j \xi_j, \gamma \right]$$

obtained by taking the bracket with the Euler vector field represents a cohomology class in $H^1(T_{\mathrm{poly}}, T_{\mathrm{poly}})$. Note that the Euler vector field does not belong to T_{poly}.

I claim that the cohomology of $C(T_{\mathrm{poly}}, T_{\mathrm{poly}})$ is generated (under cup products and up to completion) by the graph cohomology $H(\mathsf{GC})$, the wheels W_j, $j = 1, 2, \dots$, the element S as in Theorem 2 and the classes coming from sequences of polynomials as above.

However, let us "cheat" at this point and consider only the invariant part $C(T_{\mathrm{poly}}, T_{\mathrm{poly}})^{GL(\infty)}$, which is probably of higher practical interest anyways.

Proposition 3. *The cohomology of $C(T_{\mathrm{poly}}, T_{\mathrm{poly}})^{GL(\infty)}$ can be identified with the space*

$$\mathbf{S}(\mathbb{R}S \oplus \mathbb{R}T \oplus \prod_{n=1,5,9,\dots} \mathbb{R}W_n \oplus H(\mathsf{GC}))\,.$$

Here T is as in the example above and the remainder of the notation is the same as in Theorem 2.

Proof. Note that the cohomology in the Proposition is the cohomology of the subcomplex $\mathsf{fXGC}^{nt} \subset \mathsf{fXGC}$ spanned by graphs without special tadpoles. Hence it suffices to show that

$$C(T_{\rm poly}, T_{\rm poly})^{GL(\infty)} \cong \mathsf{fXGC}^{nt}\,.$$

This follows essentially from classical invariant theory for $GL(\infty)$. More precisely, let us write

$$C(T_{\rm poly}, T_{\rm poly}) = \lim_{\leftarrow} C(T^{(n)}_{\rm poly}, T_{\rm poly}).$$

Then

$$C(T_{\rm poly}, T_{\rm poly})^{GL(\infty)} = \lim_{\leftarrow} C(T^{(n)}_{\rm poly}, T_{\rm poly})^{GL(n)}.$$

Note that $C(T^{(n)}_{\rm poly}, T^{(n)}_{\rm poly})$ carries an additional bigrading (as vector space) by the input and output degrees of maps. More precisely, the space of cochains with r inputs can be written as

$$C^r(T^{(n)}_{\rm poly}, T^{(n)}_{\rm poly}) = \prod_{d_i} \bigoplus_{d_o} C^r_{d_i,d_o}(T^{(n)}_{\rm poly}, T^{(n)}_{\rm poly})$$

where $C^r_{d_i,d_o}(T^{(n)}_{\rm poly}, T^{(n)}_{\rm poly})$ is the subspace of chains that map homogeneous polynomials $\gamma_1, \dots, \gamma_r$ of total degree (number of x's and ξ's in all γ's) d_i to a polynomial of total degree d_o. Each such subspace is finite dimensional since there are only finitely many possible monomials. There is a similar additional bigrading on fXGC and the map $\mathsf{fXGC} \to T^{(n)}_{\rm poly}$ preserves the grading. In particular it follows that $\mathsf{fXGC}^{(n)} = \mathsf{fXGC}/I_n$, with I_n the kernel of the previous map, inherits this bigrading. Concretely, a graph is of bidegree (d_i, d_o) if the sum of valences of all input vertices is d_i and the valence of the output vertex is d_o. We will call $\mathsf{fXGC}^{(n)}_{d_i,d_o}$ the space spanned by such graphs.

Note furthermore that

$$C(T^{(n)}_{\rm poly}, T_{\rm poly}) = C(T^{(n)}_{\rm poly}, T^{(n)}_{\rm poly}) \hat{\otimes} \mathbb{R}[x_{n+1}, \dots, \xi_{n+1}, \dots]$$

for a suitably completed tensor product $\hat{\otimes}$ and hence

$$\begin{aligned} C(T^{(n)}_{\rm poly}, T_{\rm poly})^{GL(n)} &= C(T^{(n)}_{\rm poly}, T^{(n)}_{\rm poly})^{GL(n)} \hat{\otimes} \mathbb{R}[x_{n+1}, \dots, \xi_{n+1}, \dots] \\ &= \mathsf{fXGC}^{(n)} \hat{\otimes} \mathbb{R}[x_{n+1}, \dots, \xi_{n+1}, \dots]. \end{aligned}$$

This reduces our goal to showing that

$$\lim_{\leftarrow} \mathsf{fXGC}^{(n)} \hat{\otimes} \mathbb{R}[x_{n+1}, \dots, \xi_{n+1}, \dots] = \mathsf{fXGC}^{nt}.$$

Consider just the grading by input degrees. It allows us to reduce the statement further to showing that

$$\lim_{\leftarrow} \mathsf{fXGC}^{(n)}_{d_i} \hat{\otimes} \mathbb{R}[x_{n+1}, \dots, \xi_{n+1}, \dots] = \mathsf{fXGC}^{nt}_{d_i}$$

where the subscripts indicate that we restrict to subspaces spanned by graphs with fixed input degree (the sum of valences of all input vertices) d_i. Note that for d_i fixed and n big enough $\mathsf{fXGC}^{(n)}_{d_i} = \mathsf{fXGC}_{d_i}$. It suffices to consider terms for n big enough. Consider a graph Γ with special tadpole in fXGC_{d_i} and let Γ' be the same graph with special tadpole deleted. For polynomials $\gamma_n, \nu_n \in \mathbb{R}[x_{n+1}, \dots, \xi_{n+1}, \dots]$ the map

$$\mathsf{fXGC}^{(n)}_{d_i} \hat{\otimes} \mathbb{R}[x_{n+1}, \dots, \xi_{n+1}, \dots] \to \mathsf{fXGC}^{(n-1)}_{d_i} \hat{\otimes} \mathbb{R}[x_n, \dots, \xi_n, \dots]$$

maps

$$\Gamma \otimes \gamma_n + \Gamma' \otimes \nu_n \mapsto \Gamma \otimes \gamma_n + \Gamma' \otimes ((x^n \xi_n)\gamma_n + \nu_n)\,.$$

Hence the coefficients γ_n, ν_n of graphs Γ, Γ' for an element of the inverse limit must satisfy the equations (for $n \gg 0$)

$$\gamma_{n-1} = \gamma_n$$
$$\nu_{n-1} = (x^n \xi_n)\gamma_n + \nu_n \, .$$

These equations are impossible to satisfy unless $\gamma_n = 0$, since the element $\sum_{k \geq n} x^k \xi_k$ is not contained in $\mathbb{R}[x_n, \dots, \xi_n, \dots]$. In this case the collection $(\nu_n)_n$ has to define an element of $\lim_{\leftarrow} \mathbb{R}[x_n, \dots, \xi_n, \dots]$, i.e., the ν_n must be a constant. The claim of the Proposition follows. □

7. Discussion

In this section the results of the previous subsections are compared to similar statements in the recent work by V. Dolgushev [3]. V. Dolgushev essentially shows that the space of universal formality morphism described by Kontsevich graphs up to homtopy is a torsor over the exponential group of the zero-th cohomology of the graph complex fGC. He calls such formality morphisms *stable*. This related to the present work. In general, if we have two isomorphic objects o, o' in some category, then for some group G it is equivalent to say that (i) $\mathrm{Aut}(o) \cong G$ or that (ii) the space of isomorphisms $o \to o'$ is a G-torsor. In this section we show how to relate V. Dolgushev's notion of stability to the slightly more intrinsic one of previous sections. In particular, this yields an alternative way to prove V. Dolgushev's result, in a less combinatorial way.

7.1. **Automorphisms of T_{poly}, and stable formality morphisms.** Let $D_{\mathrm{poly}}^{(n)}$ be the space of polynomial polydifferential operators on $\mathbb{R}^n$. By the Hochschild-Kostant-Rosenberg (HKR) Theorem it is quasi-isomorphic to $T_{\mathrm{poly}}^{(n)}$ as a complex. Since taking direct limits commutes with taking cohomology, we see that the stable version

$$D_{\mathrm{poly}} = \lim_{\rightarrow} D_{\mathrm{poly}}^{(n)}$$

is quasi-isomorphic to T_{poly} as a complex. By M. Kontsevich's Formality Theorem the two spaces are also quasi-isomorphic as $(\mathsf{Lie}\{1\})_\infty$ algebras. One can consider the space M of $(\mathsf{Lie}\{1\})_\infty$ quasi-isomorphisms $T_{\mathrm{poly}} \to D_{\mathrm{poly}}$ whose unary component is the Hochschild-Kostant-Rosenberg (HKR) morphism.

Let $\mathrm{Aut}_\infty(T_{\mathrm{poly}})$ be the group of $(\mathsf{Lie}\{1\})_\infty$ automorphisms of T_{poly}. Since T_{poly} has zero differential, the unary component of any $(\mathsf{Lie}\{1\})_\infty$ automorphism is an honest $\mathsf{Lie}\{1\}$ algebra automorphism. Denote the kernel of the morphism $\mathrm{Aut}_\infty(T_{\mathrm{poly}}) \to \mathrm{Aut}(T_{\mathrm{poly}})$ by G. There is a natural right action of G on M.

We may require one of the following additional properties from our stable formality morphisms or automorphisms:

- There are natural actions of $GL(\infty)$ on T_{poly} and D_{poly}. We may ask the formality morphisms and automorphisms of T_{poly} to be equivariant with respect to this action. We obtain the space $M^{GL} \subset M$ of equivariant formality morphisms, with a right action of the group G^{GL} of equivariant $(\mathsf{Lie}\{1\})_\infty$ automorphisms of T_{poly}.
- Both T_{poly} and D_{poly} are left $\mathcal{O} = \mathbb{R}[x^1, x^2, \dots]$ modules. We may ask the components of the formality morphisms and automorphisms to be (poly)differential operators between $\mathcal{O}$ modules.
- We may combine the above and require that the components of our $(\mathsf{Lie}\{1\})_\infty$ automorphisms and formality morphisms are $GL(\infty)$ invariant polydifferential operators. We call the set of such formality morphisms $M^{pg} \subset M^{GL} \subset M$ and the group $G^{pg} \subset G^{GL} \subset G$. For example, M. Kontsevich's formality morphism is an element of M^{pg}.

7.2. **Homotopy.** We will use the following "cheap" notion of homotopy between $(\mathsf{Lie}\{1\})_\infty$ morphisms.

Definition 5. *Let G, H be $(\mathsf{Lie}\{1\})_\infty$ algebras and let $f, f' \colon G \to H$ be $(\mathsf{Lie}\{1\})_\infty$ morphisms. We say that f, f' are* homotopic on the nose *if there is an $(\mathsf{Lie}\{1\})_\infty$ morphism*

$$\alpha : G \to H[t, dt]$$

such that the restriction $\alpha|_{t=0} = f$ and $\alpha|_{t=1} = f'$.

We say that f, f' are homotopic if there are morphisms $f_1, f_2, \dots, f_n : G \to H$ for some n such that f is homotopic on the nose to f_1, f_1 is homotopic on the nose to f_2 etc., and f_n is homotopic on the nose to f'. In other words, we define the relation of homotopy as the transitve closure of the relation of homotopy on the nose.

This defines an equivalence relation on the space of morphism $G \to H$. Furthermore, let K be another $(\mathsf{Lie}\{1\})_\infty$ algebra. Suppose there are homotopic morphisms $f, f' : G \to H$ and $g, g' : H \to K$. Then clearly $g \circ f$ and $g' \circ f'$ are homotopic.

If in the previous definition G and H are copies of $T_{\rm poly}$ or $D_{\rm poly}$ and we consider morphism f, f' of one of the restricted forms of the previous subsection, we will require that the homotopies also have the restricted form.

Definition 6. *Let G, H and f, f' be as in the previous definition.*

- *Suppose the maps of complexes $G \to H$ induced by f, f' (i.e., their unary components) agree. We say that f, f' are homotopic on the nose with fixed unary component if the homotopy α may be chosen such that, in addition to the requirements of the previous definition, the restrictions at fixed t, i.e., $\alpha|_t : G \to H$, have the same unary component as f and f'.*
- *Let G, H again be copies of $T_{\rm poly}$ or $D_{\rm poly}$, and assume that f, f' are $GL(\infty)$ equivariant morphisms. Then we say that f, f' are homotopic on the nose as $GL(\infty)$ equivariant morphisms if α as above may be chosen to be $GL(\infty)$ equivariant.*
- *Restrict to G, H being copies of $T_{\rm poly}$ or $D_{\rm poly}$, and assume that f, f' are polydifferential morphisms. Then we say that f, f' are homotopic on the nose as polydifferential morphisms if α as above may be chosen to be a polydifferential morphism.*
- *etc...*

In each case we define the relation of homotopy between morphisms as the transitive closure of the relation of homotopy on the nose.

7.3. **Inversion.** Consider the Hochschild-Kostant-Rosenberg quasi-isomorphism (of complexes)

$$\phi\colon T_{\rm poly} \to D_{\rm poly}\,.$$

It has an explicit left inverse ϕ^{-1}, i.e., $\phi^{-1} \circ \phi = id_{T_{\rm poly}}$. Furthermore, there is a homotopy h given by explicit recursion relations such that $id - \phi \circ \phi^{-1} = h d_H + d_H h$ where d_H is the Hochschild differential on $D_{\rm poly}$. In addition h has the following properties:

1. ϕ^{-1} and h commute with the $GL(\infty)$ action.
2. ϕ^{-1} and h are $\mathcal{O}$-linear, i.e., degree zero differential operators.
3. ϕ^{-1} and h are a constant coefficient (degree zero) differential operators.

Explicit recursion formulas for ϕ^{-1} and for the homotopy h can be extracted from [7].

Now let $F \in M$ be a (stable) formality morphism $F\colon T_{\rm poly} \to D_{\rm poly}$. Using ϕ^{-1} and h from above one can construct (by explicit recursion formulas) a left

inverse F^{-1} to F, i.e., $F^{-1} \circ F = id_{T_{\mathrm{poly}}}$. The components of F^{-1} are all given by composing components of F with copies of ϕ, ϕ^{-1}, h and the Lie brackets. In particular, if F falls in any of the restricted classes from above ($GL(\infty)$ equivariant, polydifferential, etc.), then F^{-1} will fall in the same class because of the properties of ϕ^{-1} and h asserted above. Furthermore, the composition $F \circ F^{-1}$ is homotopic on the nose to $id_{D_{\mathrm{poly}}}$, and the homotopy is defined by suitable compositions of ϕ, ϕ^{-1}, h and the Lie brackets. In particular, if F falls in any of the restricted classes from above ($GL(\infty)$ equivariant, polydifferential, etc.) then $F \circ F^{-1}$ is homotopic on the nose to $id_{D_{\mathrm{poly}}}$ in that restricted class.

7.4. Formality morphisms up to homotopy.

Definition 7. *Let $H \subset G$ be the normal subgroup of automorphisms which are homotopic (with fixed unary part) to the identity morphism. The group of $(\mathsf{Lie}\{1\})_\infty$ automorphisms of T_{poly} up to homotopy is defined to be $\bar{G} := G/H$. Similarly we define $\bar{G}^{GL(\infty)}$ and $\bar{G}^{pg}$, as quotients of the subgroups of $GL(\infty)$ or constant coefficient polydifferential and $GL(\infty)$ equivariant morphisms, modulo the appropriate homotopy relation.*

We define $\bar{M}$ to be the quotient of M obtained by identifying homotopic formality morphisms. Similarly, we define $\bar{M}^{GL(\infty)}$ and $\bar{M}^{pg}$.

The right actions of G on M descends to the quotient, so that we obtain an action of $\bar{G}$ on $\bar{M}$. Similarly $\bar{G}^{GL(\infty)}$ and $\bar{G}^{pg}$ act on $\bar{M}^{GL(\infty)}$ and $\bar{M}^{pg}$ (from the right).

Lemma 2. *$\bar{M}$ is a $\bar{G}$-torsor, $\bar{M}^{GL(\infty)}$ is a $\bar{G}^{GL(\infty)}$-torsor and $\bar{M}^{pg}$ is a $\bar{G}^{pg}$-torsor.*

Proof. It is a purely formal statement (and standard fact), but let us do the argument for $\bar{M}$ and $\bar{G}$.

Faithfulness: Let $f \in G$, $F \in M$ and suppose Ff is homotopic to F. Then we need to show that f is homotopic to the identity. Applying F^{-1} we see that $F^{-1}Ff$ is homotopic to $F^{-1}F$. But $F^{-1}F$ is homotopic to the identity and hence f is, too, being homotopic to $F^{-1}Ff$.

Transitivity: Let $F, F' \in M$. Then we need to show that there is an f such that Ff is homotopic to F'. Take $f = F^{-1}F'$, so we need to show that $FF^{-1}F'$ is homotopic to F'. It is sufficient to show that $F^{-1}FF^{-1}F'$ is homotopic to $F^{-1}F'$. But this is clear since both morphisms are equal. □

7.5. Reduction to $\mathsf{Lie}\{1\}$ algebra cohomology.

G is a pro-nilpotent group and thus the exponential group of its Lie algebra. The Lie algebra is the space of $(\mathsf{Lie}\{1\})_\infty$ derivations of T_{poly}, with zero unary component. This space in turn can be seen as the closed degree zero elements in $C(T_{\mathrm{poly}}, T_{\mathrm{poly}})$, with vanishing unary (and zero-ary) part. An element $\exp(x) \in G$, with x a degree zero cocyle in $C(T_{\mathrm{poly}}, T_{\mathrm{poly}})$, is homotopic to the identity iff x is exact. Hence the Lie algebra of $H \subset G$ (with H as above) is the exponential group of the Lie algebra of degree zero cocycles in $C(T_{\mathrm{poly}}, T_{\mathrm{poly}})$, with vanishing unary and zero-ary part. The Lie algebra of $\bar{G}$ is hence given by the subspace of $H^0(T_{\mathrm{poly}}, T_{\mathrm{poly}})$ of elements without unary or zero-ary part. Let us denote this subspace by $H^0(T_{\mathrm{poly}}, T_{\mathrm{poly}})'$. Hence, using Lemma 2 we see that $\bar{M}$ is a torsor over the exponential group of $H^0(T_{\mathrm{poly}}, T_{\mathrm{poly}})'$. Similar reasoning applies for the $GL(\infty)$ equivariant or constant coefficient polydifferential cases, so we obtain the following statements:

- The space $\bar{M}^{GL(\infty)}$ is a torsor over $H^0(C(T_{\mathrm{poly}}, T_{\mathrm{poly}})^{GL(\infty)})'$, where the $'$ again means that we restrict to those derivations having vanishing unary and zero-ary part.

- The space $\bar{M}^{cg}$ is a torsor over the exponential group of the Lie algebra $H^0(C_{polydiff}(T_{\rm poly}, T_{\rm poly})^{GL(\infty)})'$. Here $C_{polydiff}(T_{\rm poly}, T_{\rm poly}) \subset C(T_{\rm poly}, T_{\rm poly})$ are those chains given by $\mathcal{O}$-polydifferential operators, and the $'$ again means that we require vanishing unary and zero-ary part.

7.6. **Reduction to graph cohomology.** As we saw above (or by definition), $C(T_{\rm poly}, T_{\rm poly})^{GL(\infty)} \cong \mathsf{fXGC}^{nt}$. Is is not hard to see that the graphs that give rise to polydifferential morphism are those in the image of fGC. Hence, combining the statements above we obtain the following result:

- The space $\bar{M}^{GL(\infty)}$ is a torsor over the exponential group of $H^0(\mathsf{fXGC})'$.
- The space $\bar{M}^{pg}$ is a torsor over the exponential group of $H^0(\mathsf{fGC})$.

In the first case the $'$ again means that we restrict to those elements having vanishing unary and zero-ary part. In other words, we forbid cohomology classes represented by graphs with less than 2 input vertices. In the second case this is not necessary since there are no such classes.

7.7. **Comparison to** [3]**.** Let us compare this to a recent result of V. Dolgushev [3]. He defines a notion of *stable formality morphism* as a universal formality morphism given by Kontsevich graphs, whose unary component is the HKR morphism. Let us call the set of such formality morphisms M'. There is also a natural notion of homotopy on morphisms in M'. Let us denote by $\bar{M}'$ the quotient obtained by identifying homotopic morphisms (in his sense). V. Dolgushev then shows that $\bar{M}'$ is a torsor for the exponential group of the cohomology of the full graph complex $H^0(\mathsf{fGC})$. This relates to the present work as follows. By adapting the arguments in the proof of Proposition 3 and those of the previous subsection, one sees that the formality morphisms in M^{pg} are precisely those expressible via Kontsevich graphs, i.e., $M' = M^{pg}$. Given the statements of the previous subsection we can hence recover the result of [3] along a different (less combinatorial) route.

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